On the Curvature of Space

By
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Abstract

In deriving a cosmological model from his general theory of relativity, Einstein somewhat arbitrarily opted for a static universe. The mathematical consequence of this decision was a nonzero value for one of the constants of integration, the so-called cosmological constant, Λ. From a Newtonian analogue, Λ can be viewed as representing a repulsive force that increases with distance and that keeps the universe from collapsing under gravitational attraction.

In the following selection, the Russian mathematician Aleksandr Friedman considers non-static models for the first time. By treating the spatial curvature of the universe as a function of time, he shows the possibility of non-stationary worlds with positive and negative curvature. These dynamic world models became especially important several years later when the universe of galaxies was found to be expanding. Although Friedman's non-static cosmology was for some years overlooked by astronomers, Einstein noticed this paper and within a few months issued a one-paragraph critique in the same journal, only to retract his objection early in 1923.

Friedman begins with the general idea that at any given instant of time the cosmological model represents a space of positive spatial curvature $R(t)$. If R is independent of time, then the stationary world models of Einstein and Wilhelm de Sitter follow. If $R(t)$ depends only on the time variable, then a variety of monotonically expanding or periodically oscillating models result, depending on the value chosen for Λ. Friedman notes that with $\Lambda = 0$, there follows an oscillating model whose period depends on the total mass of the universe.

In a second paper Friedman considers models with negative curvature. He finds a non-stationary world with negative spatial curvature and positive matter density, but no static model. Friedman also notes in this later contribution that Einstein's field equations do not suffice to extract a conclusion about the finiteness of space without some supplementary assumptions.

1 Introduction

1. In their well-known works on general cosmological questions, EINSTEIN, [5] and DE SITTER, [1, 2, 3], arrive at two possible types of universe: EINSTEIN obtains the so-called cylindrical world, in which space[1] possesses a constant curvature independent of time and in which the radius of curvature is connected with the total mass of matter existing in space. DE SITTER obtains a spherical world in which not only space but also the world can be spoken of, in a certain sense, as a world of constant curvature, [7]. In doing so, certain assumptions about the matter tensor are made by both EINSTEIN and DE SITTER; these correspond to the incoherence of matter and its being relatively at rest, e.g. the velocity of matter is assumed to be sufficiently small in comparison with the fundamental velocity[2] – the velocity of light.

The goal of this notice is, first, the derivation of the cylindrical and spherical worlds (as special cases) from some general assumptions and, second, the proof of the possibility of a world whose spatial curvature is constant with respect to three coordinates that are permissible spatial coordinates and that depend on time, e.g. on the fourth (time) coordinate. This new type is, as far as its remaining properties are concerned, an analogue of the EINSTEINian cylindrical universe.

2 The Assumptions

2. The assumptions on which we shall base our considerations break down into two classes. To the first class belong assumptions that coincide well with the assumptions of EINSTEIN and DE SITTER. They refer to the equations that the gravitational potentials satisfy and to the state and motion of matter. To the second class belong assumptions about the general, so-to-speak geometric, character of the world. From our hypothesis the cylindrical world of EINSTEIN and the spherical world of DE SITTER follow as special cases.

The assumptions of the first class are the following:

a.1 – The gravitational potentials satisfy the EINSTEIN system of equations with the cosmological term, which we may also set equal to zero:

$$R_{ik} - \frac{1}{2} \cdot g_{ik} \cdot \overline{R} + \lambda g_{ik} = -\kappa \cdot T_{ik} \quad (i, k = 1, 2, 3, 4)\,. \tag{A}$$

Here the g_{ik} are the gravitational potentials, T_{ik} is the matter tensor (energy-momentum tensor), κ is a constant, $\overline{R} = g^{ik} R_{ik}$, (the cosmological constant

[1]By “space” we mean here a space that is described by a manifold of three dimensions; The “world” corresponds to a manifold of four dimensions.
[2]Regarding this term see the book by EDDINGTON, [4]

is denoted by λ) and R_{ik} is determined by the equations

$$R_{ik} = \frac{\partial^2 \log(\sqrt{g})}{\partial x_i\, \partial x_k} - \frac{\partial \log(\sqrt{g})}{\partial x_\sigma} \begin{Bmatrix} i\ k \\ \sigma \end{Bmatrix} - \frac{\partial}{\partial x_\sigma} \begin{Bmatrix} i\ k \\ \sigma \end{Bmatrix} + \begin{Bmatrix} i\ \alpha \\ \sigma \end{Bmatrix} \begin{Bmatrix} k\ \sigma \\ \alpha \end{Bmatrix}, \tag{B}$$

where the $x_i (i = 1, 2, 3, 4)$ are world coordinates and $\begin{Bmatrix} i\ k \\ l \end{Bmatrix}$ are the CHRISTOFFEL symbols of the second kind[3].

b.1 – The matter is incoherent and relatively at rest. Stated less strongly, the relative velocities of matter are vanishingly small in comparison with the velocity of light. In consequence of these assumptions, the matter tensor is given by the equations

$$\left.\begin{aligned} T_{ik} &= 0 \text{ for } i \text{ and } k \neq 4\,, \\ T_{44} &= c^2 \cdot \rho \cdot g_{44}\,. \end{aligned}\right\} \tag{C}$$

Here ρ is the density of matter and c is the fundamental velocity. Moreover, the world coordinates are divided into three spatial coordinates x_1, x_2, x_3 and the time coordinate x_4.

The assumptions of the second class are the following:

a.2 – After distribution of the three spatial coordinates x_1, x_2, x_3 we have a space of constant curvature, which, however, may depend on x_4, the time coordinate. The interval[4] $\mathrm{d}s$, determined by $\mathrm{d}s^2 = g_{ik}\,\mathrm{d}x_1\,\mathrm{d}x_k$ can be brought into the following form by the introduction of suitable spatial coordinates:

$$\begin{aligned} \mathrm{d}s^2 = R^2 \cdot &\left\{\mathrm{d}x_1^2 + \sin^2(x_1)\,\mathrm{d}x_2^2 + \sin^2(x_1)\cdot\sin^2(x_2)\,\mathrm{d}x_3^2\right\} \\ &+ 2g_{14}\,\mathrm{d}x_1\,\mathrm{d}x_4 + 2g_{24}\,\mathrm{d}x_2\,\mathrm{d}x_4 + 2g_{34}\,\mathrm{d}x_3\,\mathrm{d}x_4 + g_{44}\,\mathrm{d}x_4^2\,. \end{aligned}$$

Here R depends only on x_4 and it is proportional to the radius of curvature of space, which may therefore change with time.

b.2 – In the expression for $\mathrm{d}s^2$, the g_{14}, g_{24}, g_{34} can be made to vanish by a suitable choice of the time coordinate. In brief, time is orthogonal to space. It seems to me that no physical or philosophical grounds can be given for the second assumption. It serves exclusively to simplify the calculation. One must still notice that the worlds of EINSTEIN and DE SITTER are contained in our assumptions as special cases.

In consequence of assumptions **1.** and **2.**, $\mathrm{d}s^2$ can be brought into the form

$$\mathrm{d}s^2 = R^2 \cdot \left\{\mathrm{d}x_1^2 + \sin^2(x_1)\,\mathrm{d}x_2^2 + \sin^2(x_1)\sin^2(x_2)\,\mathrm{d}x_3^2\right\} + M^2 \cdot \mathrm{d}x_4^2\,, \tag{D}$$

[3]The sign of R_{ik} and $\overline{R}$ differs here from the usual convention.
[4]Also line element, see `https://en.wikipedia.org/wiki/Line_element`

where R is a function of x_4 and M depends, in the general case, on all four world coordinates. The EINSTEIN universe is obtained if one replaces $R^2 = -\frac{R^2}{c^2}$ in Equation D and if one also sets $M = 1$, whereby R signifies the constant (independent of x_4) radius of curvature of space.

$$\mathrm{d}\tau^2 = -\frac{R^2}{c^2} \cdot \left\{\mathrm{d}x_1^2 + \sin^2(x_1)\,\mathrm{d}x_2^2 + \sin^2(x_1)\sin^2(x_2)\,\mathrm{d}x_3^2\right\} + \mathrm{d}x_4^2\,. \qquad (\mathrm{D}_1)$$

The universe of DE SITTER is obtained if one replaces R^2 by $-\frac{R^2}{c^2}$ and M by $\cos(x_1)$ in Equation D[5]

$$\begin{aligned}\mathrm{d}\tau^2 = &-\frac{R^2}{c^2} \cdot \left\{\mathrm{d}x_1^2 + \sin^2(x_1)\,\mathrm{d}x_2^2 + \sin^2(x_1)\sin^2(x_2)\,\mathrm{d}x_3^2\right\} \\ &+ \cos^2(x_1)\cdot \mathrm{d}x_4^2\,. \end{aligned} \qquad (\mathrm{D}_2)$$

3. – Now we must still strike an agreement about the boundaries within which the world coordinates are confined, e.g. what points of the 4-dimensional manifold we will treat as different. Without engaging in a more detailed motivation, we shall assume that the spatial coordinates are confined to the following intervals: x_1 in the interval $(0, \pi)$, x_2 in the interval $(0, \pi)$, and x_3 in the interval $(0, 2\pi)$. With respect to the time coordinate we make, for the present, no restricting assumptions, but we shall consider this question further below.

3 Curved Space-Time in Stationary and Non-stationary Worlds

A. – From the assumptions according to Equation C and Equation D it follows, if one sets $i = 1, 2, 3$ and $k = 4$ in Equation A, that

$$R'(x_4)\cdot\frac{\partial M}{\partial x_1} = R'(x_4)\cdot\frac{\partial M}{\partial x_2} = R'(x_4)\cdot\frac{\partial M}{\partial x_3} = 0\,.$$

Two cases arise.

1. $R'(x_4) = 0$, R is independent of x_4. We shall designate this world as a *stationary world.*
2. $R'(x_4) \neq 0$, M depends only on x_4. This shall be called a *non-stationary world.*

[5]The ds, which is taken to have the dimension of time, we designate dτ; then the constant κ has the dimension Length per Mass and in c.g.s. units equals 1.87×10^{-27}. See [8].

We consider, first, the stationary world and write the Equation A for $i, k = 1, 2, 3$ and moreover $i \neq k$. Then we obtain the following system of formulæ:

$$\frac{\partial^2 M}{\partial x_1 \, \partial x_2} - \cot(x_1) \cdot \frac{\partial M}{\partial x_2} = 0\,,$$
$$\frac{\partial^2 M}{\partial x_1 \, \partial x_3} - \cot(x_1) \cdot \frac{\partial M}{\partial x_3} = 0\,,$$
$$\frac{\partial^2 M}{\partial x_2 \, \partial x_3} - \cot(x_2) \cdot \frac{\partial M}{\partial x_3} = 0\,.$$

The integration of these equations yields the following expression for M:

$$M = A(x_3, x_4) \cdot \sin(x_1) \cdot \sin(x_2) + B(x_2, x_4) \cdot \sin(x_1) + C(x_1, x_4)\,, \quad (1)$$

where A, B, C are arbitrary functions of their arguments. If we solve Equation A for R_{ik}, and eliminate the unknown density ρ[6] from the still-unused equations, we obtain, if we insert for M Equation 1, the following two possibilities for M after tedious, but elementary calculations:

$$M = M_0 = \text{const.}\,, \quad (2)$$
$$M = (A_0 x_4 + B_0) \cdot \cos(x_1)\,, \quad (3)$$

where M, A_0 and B_0 are constants.

If M is equal to a constant, then the stationary world is the cylindrical world. Here it is advantageous to work with the gravitational potentials of Equation D_1. If we determine the density and the quantity λ, then the well-known result of Einstein is obtained:

$$\lambda = \frac{c^2}{R^2}\,,$$
$$\rho = \frac{2}{\kappa \cdot R^2}\,,$$
$$\overline{M} = \frac{4\pi^2}{\kappa} \cdot R\,,$$

where $\overline{M}$ denotes the total mass of space.

In the second possible case, when $\overline{M}$ is given by Equation 3, we get, by means of a judicious transformation[7] of x_4, the spherical world of de Sitter in which $M = \cos(x_1)$; with the help of Equation D_2 we obtain the relations of de Sitter:

$$\lambda = \frac{3c^2}{R^2}\,,$$
$$\rho = 0\,,$$
$$\overline{M} = 0\,.$$

[6]The density in our case is an unknown function of the world coordinates x_1, x_2, x_3, x_4.
[7]This transformation is given by the formula $\mathrm{d}\overline{x}_4 = \sqrt{A_0 x_4 + B_0}\,\mathrm{d}x_4$

We thus have the following result: *the stationary world is either the* EINSTEIN *cylindrical world or the* DE SITTER *spherical world.*

B. – We now want to consider the non-stationary world. M is now a function of x_4. By an appropriate choice of x_4 one can obtain $M = 1$, without loss of generality. In order to couple to our customary presentation, we give $\mathrm{d}s^2$ a form that is analogous to Equation D_1 and Equation D_2:

$$\mathrm{d}\tau^2 = -\frac{R^2(x_4)}{c^2} \cdot \left\{\mathrm{d}x_1^2 + \sin^2(x_1)\,\mathrm{d}x_2^2 + \sin^2(x_1)\cdot\sin^2(x_2)\,\mathrm{d}x_3^2\right\} + \mathrm{d}x_4^2\,. \quad (\mathrm{D}_3)$$

Our task is now the determination of R and ρ from the Equation A. It is clear that the Equation A with different indices yield nothing. The Equation A for $i = k = 1, 2, 3$ give the relation

$$\frac{R'^2}{R^2} + \frac{2\cdot RR''}{R^2} + \frac{c^2}{R^2} - \lambda = 0\,. \tag{4}$$

The Equation A with $i = k = 4$ yields the relation

$$\frac{3\cdot R'^2}{R^2} + \frac{3c^2}{R^2} - \lambda = \kappa c^2 \rho\,, \tag{5}$$

with

$$R' = \frac{\mathrm{d}R}{\mathrm{d}x_4} \quad \text{and} \quad R'' = \frac{\mathrm{d}^2 R}{\mathrm{d}x_4^2}\,.$$

Because $R' \neq 0$, the integration of Equation 4, if we write t for x_4, gives the following equation:

$$\frac{1}{c^2}\cdot\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{A - R + \frac{\lambda}{3c^2}\cdot R^3}{R}\,, \tag{6}$$

where A is an arbitrary constant. From this equation, we obtain R through the inversion of an elliptic integral, e.g. through the solution for R of the equation

$$t = \frac{1}{c}\cdot\int_a^R \sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2}\cdot x^3}}\,\mathrm{d}x + B\,, \tag{7}$$

in which B and a are constants. Attention must still be paid to the usual conditions about sign variation in the square root. The mass density, ρ may be determined from equation

$$\rho = \frac{3\cdot A}{\kappa\cdot R^3}\,. \tag{8}$$

The constant A is expressed in terms of the total mass of space $\overline{M}$ in the following way:

$$A = \frac{\kappa\cdot\overline{M}}{6\cdot\pi^2}\,. \tag{9}$$

If $\overline{M}$ is positive, then A will also be positive.

C. – We must base the consideration of the non-stationary world on Equation 6 and Equation 7. The quantity λ is not determined by these equations. We shall postulate that it can have an arbitrary value. We now determine these values of the variable x, for which the square root of Equation 7 changes its sign. If we restrict our consideration to positive radii of curvature, it will suffice to consider the interval $(0, \infty)$ for x and in this interval the values of x that make the radicand equal to zero or ∞. One value of x for which the square root in Equation 7 equals 0 is $x = 0$. The remaining values of x, for which the square root in Equation 7 changes sign, are given by the positive roots of the equation

$$A - x + \frac{\lambda}{3c^2} \cdot x^3 = 0\,.$$

We denote $\frac{\lambda}{3c^2}$ by y and consider the family of third degree curves in the (x, y) plane.

$$y \cdot x^3 - x + A = 0\,. \tag{10}$$

Here A is the parameter of the family of curves, which varies over the interval $(0, \infty)$. A curve of the system cuts the x-axis at the point $x = A$, $y = 0$ and has a maximum at the point

$$x = \frac{3 \cdot A}{2}$$
$$y = \frac{4}{27 \cdot A^2}\,.$$

From Figure 1 it is obvious that the equation $A - x + \frac{\lambda}{3c^2} \cdot x^3 = 0$ has a positive root x_0 in the interval $(0, A)$ for negative A. If one considers x_0 as a function of λ and A, then

$$x_0 = \Theta(\lambda, A)\,,$$

one finds that Θ is an increasing function of λ and of A. If λ is in the interval $\left(0, \frac{4}{9} \cdot \frac{c^2}{A^2}\right)$, the equation has two positive roots $x_0 = \Theta(\lambda, A)$ and $x_0' = \Phi(\lambda, A)$, where x_0 is the root in the interval $\left(A, \frac{3A}{2}\right)$ and x_0' is in the interval $\left(\frac{3A}{2}, \infty\right)$. $\Theta(\lambda, A)$ is an increasing function of λ and A, whereas $\Phi(\lambda, A)$ is a decreasing function of λ and A. Finally, if λ is bigger than $\frac{4}{9} \cdot \frac{c^2}{A^2}$, then the equation has no positive roots.

Let us now pass on to a discussion of Equation 7 taking into consideration the following remark: Let the radius of curvature equal R_0 for $t = t_0$. The sign of the square root in Equation 7 is positive or negative for $t = t_0$ depending on whether the radius of curvature is increasing or decreasing for $t = t_0$. By replacing t with $-t$ if necessary, we can always make the square root positive, i.e. by choosing the time we can always ensure that the radius

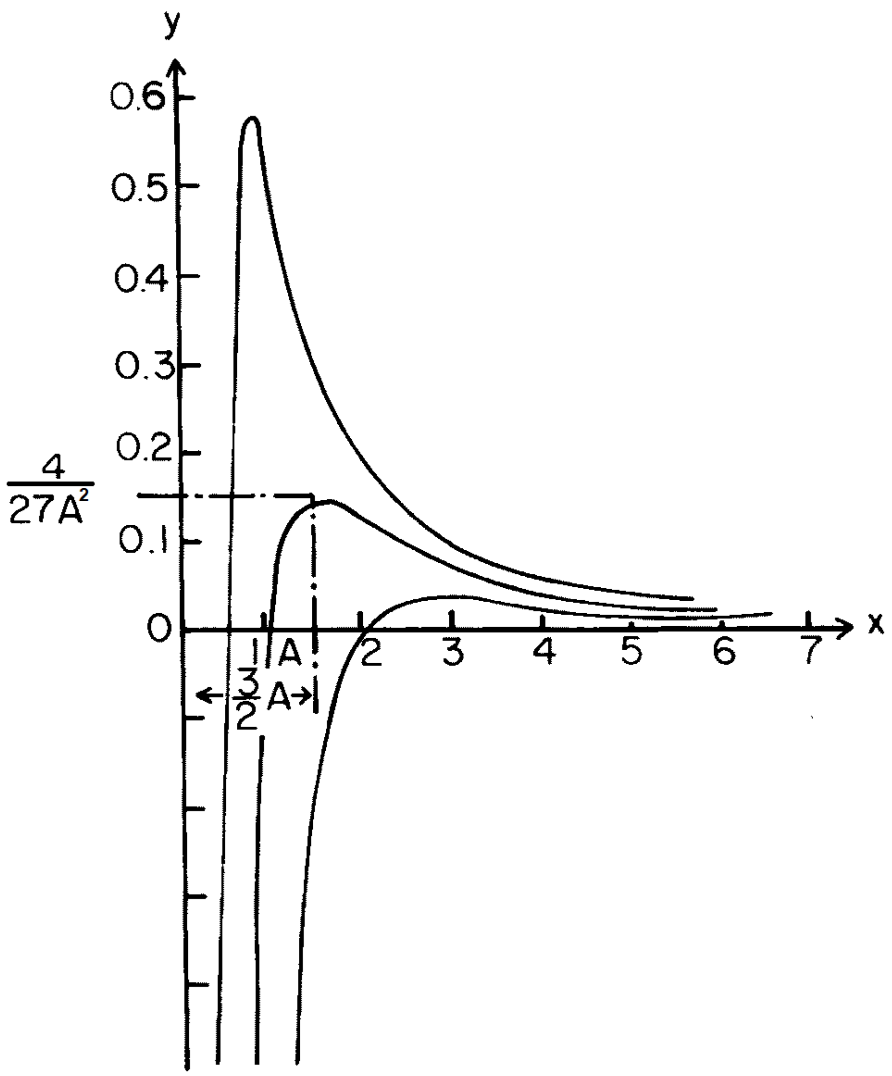

Figure 1: A plot in the (x, y) plane of the curves satisfying the non-stationary world equation $y \cdot x^3 - x + A = 0$ where $y = \frac{\lambda}{3c^2}$ is the reciprocal of the radius R and A is a constant which is proportional to the total mass of the universe

of curvature for $t = t_0$ increases with increasing time.

D. – We consider first the case $\lambda > \frac{4}{9} \cdot \frac{c^2}{A^2}$, e.g. the case in which the equation $A - x + \frac{\lambda}{3c^2} \cdot x^3 = 0$ has no positive roots. Equation 7 can then be written in the form

$$t - t_0 = \frac{1}{c} \cdot \int_{R_0}^{R} \sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2} \cdot x^3}}\, \mathrm{d}x\,, \tag{11}$$

where, in consequence of our remark, the square root is always positive. From that, it follows that R *is an increasing function of* t. The positive initial value R_0 is free of any restriction.

Since the radius of curvature is not allowed to be smaller than zero, it must reach the value zero as time t decreases from R_0 at the moment t'. We will call the time when R increases from 0 to R_0 the time since the creation of the world[8]. This time, t', is given by

$$t' = \frac{1}{c} \cdot \int_{0}^{R_0} \sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2} \cdot x^3}}\, \mathrm{d}x\,. \tag{12}$$

We denote the world under consideration as a *monotonic world of the first kind.*

The time since the creation of (the monotonic) world (of the first kind), considered as a function of R_0, A, λ, has the following properties:

1. It increases with increasing R_0,
2. It decreases if A increases, e.g. if the mass in space is increased,
3. It decreases if λ increases.

If $A > \frac{2}{3}R_0$, then for an arbitrary λ the time elapsed since the creation of the world is finite. If $A \leqslant \frac{2}{3}R_0$, then a value of $\lambda = \lambda_1 = \frac{4c^2}{9A^2}$ can always be found such that as λ approaches this value, the time since the creation of the world increases without limit.

E. – Now let λ lie in the interval $\left(0, \frac{4c^2}{9A^2}\right)$; then the initial value of the radius of curvature can lie in one of the intervals

$$(0, x_0)\,, \quad (x_0, x_0')\,, \text{ or } (x_0', \infty)\,.$$

If R_0 falls in the interval (x_0, x_0'), then the square root in Equation 7 is imaginary. A space with this initial curvature is impossible.

[8]The time since the creation of the world is the time that has elapsed from the moment when space was a point ($R = 0$) to the present state ($R = R_0$); this time can also be infinite.

We devote the next section to the case where R_0 lies in the interval $(0, x_0)$. Here we also consider the third case: $R_0 > x'_0$ or $R_0 > \Phi(\lambda, A)$. Through considerations that are analogous to the preceding ones, it can be shown that R is an increasing function of time, whereby R can begin with the value $x'_0 = \Phi(\lambda, A)$. The time that has elapsed from the moment when $R = x'_0$ to the moment that corresponds to $R = R_0$, we again call the time since the creation of the world. Let it be t'; then

$$t' = \frac{1}{c} \cdot \int_{x'_0}^{R_0} \sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2} \cdot x^3}}\, \mathrm{d}x\,. \tag{13}$$

We call this world a *monotonic world of the second kind.*

F. – We now consider the case that λ falls between the limits $(-\infty, 0)$. If $R_0 > x_0 = \Theta(\lambda, A)$, the square root in Equation 7 becomes imaginary, and the space with this R_0 is impossible. If $R_0 < x_0$, the considered case is identical with that which we have left aside in the preceding sections. We therefore assume that λ lies in the interval $\left(-\infty, \frac{4}{9} \cdot \frac{c^2}{A^2}\right)$ and $R_0 < x_0$. By means of a well-known argument[9] one can now show that R becomes a periodic function of t with the period t_π, which we name the *world period*; t_π is given by the formula

$$t_\pi = \frac{2}{c} \cdot \int_{0}^{x_0} \sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2} \cdot x^3}}\, \mathrm{d}x\,. \tag{14}$$

The radius of curvature varies between 0 and x_0. We shall call this universe the *periodic world.* The period of the periodic world increases if we increase λ and tends to infinity if λ tends to the value $\lambda_1 = \frac{4}{9} \cdot \frac{c^2}{A^2}$.

For small λ, the period is represented by the approximate formula

$$t_\pi = \frac{\pi \cdot A}{c}\,. \tag{15}$$

With reference to the periodic world two points of view are possible: We count two events as coincident if their spatial coordinates coincide and the difference of time coordinate is an integral multiple of the period, so that the radius of curvature grows from 0 to x_0 and thereafter decreases to the value 0. The time of world existence is finite. On the other hand, if the time varies between $-\infty$ and $+\infty$ (e.g. if we consider two events as coincident only when not only their spatial but also their world coordinates coincide), we come to a real periodicity of the space curvature.

[9]See, e.g. [9], [6]. In our case, the considerations of these authors must be modified accordingly; However, in our case the periodicity is determined by elementary considerations.

G. – Our knowledge is completely insufficient to carry out numerical calculations and to distinguish which world our universe is. It is possible that the causality problem and the problem of centrifugal force will illuminate these questions. It remains to note that the "cosmological" magnitude λ remains undetermined in our formulæ, because it is a superfluous constant in the problem. Possibly electrodynamic considerations can lead to its evaluation. It we set $\lambda = 0$ and $M = 5 \times 10^{21}$ solar masses, the world period becomes of the order 10 billion years. However, these numbers are valid only as an illustration of our calculation.

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